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A DEGREE SELECTION METHOD OF MATRIX CONDENSATIONS FOR EIGENVALUE PROBLEMS

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A new and automatic degree selection technique based on the approximate modal energy has been derived and developed for matrix condensations in this paper. The method is used to condense the number of degrees of the matrix when dealing with eigenvalue problems. By defining a new basis in the vicinity of the original space, the individual modal energy gradients can be evaluated. The primary degrees of freedom are then determined according to the variation of the energies in the neighborhood. In case the energy variation of a degree tends to increase in that neighborhood, the degree is classified as secondary since it relatively provides energy to the nodes nearby. On the other hand, if the energy variation is decreasing, then it is primary. All the classification criteria are finally mapped to one parameter, which is called the index of classification. That is, by examining the magnitude of the index of classification, one is able to determine the primary and secondary degrees. The new selection method is demonstrated and verified by a well-known cantilever beam problem in addition to the error bound estimation.

1. INTRODUCTION

Applying the finite element method (FEM), the determination of eigenvalues together with their corresponding eigenvectors plays an important role in the study of dynamic behaviors of structural systems. This is mainly because a sufficient number of eigenvalues and vectors define fairly well the characteristics of a dynamic problem. However, the high order of matrices obtained by a finite element discretization makes it impractical in the speed of computation. As a consequence, a special treatment of this type of eigenproblems so that the lowest eigenfrequencies can be accurately and effectively computed is essential.

The difficulties are usually overcome by the so-called condensation (or reduction) techniques, which reduce the order of system matrices by picking out several primary (or principal) degrees of freedom (d.o.f.s) and excluding auxiliary (or secondary) ones. The concept of this matrix condensation is a well-known procedure [1, 2] and may be regarded as Gaussian elimination of displacement in the matrix form in a static problem. The same concept was first applied to dynamic problems by Irons [3], later by Guyan [4] and is referred as the "static condensation" [5] or "Guyan reduction." The accuracy was reported well in the case of a lumped mass analysis, when elimination is done only to those massless degrees in the mass matrix [4, 6]. On the other hand, however, the accuracy depends to a large extent on the experience of the analyst to select or to redistribute the mass appropriately, and that the accuracy is hardly assessed.

A slightly advanced condensation technique is "dynamic condensation" or "mass condensation" which is derived by assuming that the virtual work done by inertial forces

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in the original system is equal to that done by the inertial forces in the reduced system [6]. The poor accuracy of static condensation is improved by the dynamic condensation technique [6–8] by taking into account the inertial effects of secondary degrees to the final form. Using the dynamic condensation, in fact the accuracy of dynamic condensation is reached mainly by distributing right mass to the corresponding degrees when a consistent mass form is used in the FE formulation. The so-called "modified dynamic condensation method" proposed by Paz [9] was reported to have better accuracy and reasonably fast in computation if elemental elimination process has been done to the original system matrices. Unfortunately, the technique still largely relies on the user's experiences to select the primary d.o.f.s.

In order to automatically select the principal co-ordinates, Grinenko and Mokeev [10] tried to develop a pure machine-dependent algorithm. The algorithm was named "dynamic frequency condensation," in which a very strong condition is assumed within the condensation ranges. This condensation method was also found [11] too time consuming in real application although the main shortcoming of the dynamic condensation was removed. Motivated by this, the author has presented a new reduction with a new mass corrector [11,12] a few years ago by using the similarity transformation. However, more and more calculations have revealed that such successive transformations are hard to preserve the relation back to the original space. And thus it limits in use. In order to overcome this difficulty, this report presents a new concept to select the primary d.o.f.s.

2. DERIVATION OF THE METHOD

In the dynamic analysis of a structural system, it is very common to utilize a FEM. For example, the equation of motion for a structure can be represented in the matrix form as

$$\mathscr{M}\ddot{\mathbf{x}} + \mathbf{r}(\dot{\mathbf{x}}, \mathbf{x}) = \mathbf{f}(t), \tag{1a}$$

where $\mathscr{M}$ is the mass matrix, $\mathbf{r}$ the damped restoring force vector and $\mathbf{f}$ the external excitation force vector. The restoring force vector $\mathbf{r}(.)$ may be linear or non-linear depending on the nature of structural systems. In case of non-linear ones, it may be linearized by using the so-called linearization methods, e.g., [13] for most problems. Thus, it is reasonable to restrict the consideration to the linear one, or equation (1a) has the form

$$\mathscr{M}\ddot{\mathbf{x}} + \mathscr{C}\dot{\mathbf{x}} + \mathscr{K}\mathbf{x} = \mathbf{f}(t), \tag{1b}$$

where $\mathscr{C}$ and $\mathscr{K}$ are the classical damping and stiffness matrices respectively. On the other hand, if the state vector $\mathbf{u}$(t) is defined as

$$\mathbf{u}^{\mathrm{T}} = \{\mathbf{x}^{\mathrm{T}}, \dot{\mathbf{x}}^{\mathrm{T}}\} \tag{2}$$

then equation (1) takes the first order form

$$M\,\dot{\mathbf{u}} + K\,\mathbf{u} = \begin{Bmatrix} \mathbf{f}(t) \\ 0 \end{Bmatrix}, \tag{3}$$

in which

$$M = \begin{bmatrix} \mathscr{C} & \mathscr{M} \\ \mathscr{M} & 0 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} \mathscr{K} & 0 \\ 0 & -\mathscr{M} \end{bmatrix}. \tag{4}$$

All these $\mathscr{M}$, $\mathscr{C}$ and $\mathscr{K}$ matrices can be obtained by the FE assemblage. Basically, augmented matrices M and K still, respectively, keep the properties of mass and stiffness of the system.

The equation of motion may be numerically solved either in real mode from equation (1) or in complex made from equation (3). In general, modal approximation methods are most widely adopted for the estimation of the system responses. That is, one assumes that the responses due to the force vector $\mathbf{f}(t)$ of certain frequency will have large participation factors for frequencies nearby certain natural frequencies of the structure. In other words, the true system responses may be approximated by a new basis of a smaller dimension. The basis certainly is a subset of the original modal vectors, while the quality of the approximation depends mainly on: (a) how many modal vectors are included to the subset to form this new basis; and (b) how the subset basis can represent the original modal ones. There exist many studies concerning first part. Papers or reports on the mode superposition or acceleration methods are the best examples. For the second part, not as popular as its counter part, it did not attract too much attention in this area. The present report will concentrate its discussion on this part.

It is clear that all the modal methods start with an eigenvalue problem. That is, based either on equation (1) or (3), one is to consider the generalized eigenvalue problem in the matrix form

$$K\Phi = M\Phi\Lambda \tag{5}$$

with conditions

$$\Phi^{\mathrm{T}}K\Phi = \Lambda \quad \text{and} \quad \Phi^{\mathrm{T}}M\Phi = I, \tag{6}$$

where Λ is a diagonal matrix containing all system eigenvalues λ or ω^2, and the columns in Φ are the corresponding eigenvectors or the mode-shapes. The stiffness matrix K is positive definite, may be symmetric or non-symmetric, and has order n. On the other hand, the matrix M may be positive definite or semi-definite depending on the mass matrix $\mathscr{M}$. However, the mass matrix is assumed to be positive definite since it can be reduced to that by a traditional static condensation or a shift [14]. From the physical point of view or from the properties of the external excitations, some d.o.f.s in the FE assemblage matrices are considered to have more weight to the final responses than the others. Equation (5) may be written in the partitioned matrix form for an arbitrary eigenpair (λ, ϕ) as

$$\begin{bmatrix} K_{ss} & K_{sr} \\ K_{rs} & K_{rr} \end{bmatrix} \begin{Bmatrix} \phi_s \\ \phi_r \end{Bmatrix} = \lambda \begin{bmatrix} M_{ss} & M_{sr} \\ M_{rs} & M_{rr} \end{bmatrix} \begin{Bmatrix} \phi_s \\ \phi_r \end{Bmatrix}, \tag{7}$$

where the subscript r denotes the principal or primary co-ordinates which are comparably significant and account for a large error if they are neglected, and the subscript s stand for the secondary co-ordinates and in case they are dropped may not cause too much error. In order to write in terms of the format equation (7), both matrices of M and K may need to be re-arranged so that the primary co-ordinates cluster to the lower part of the matrix. Notice also that the secondary co-ordinates can be regarded as internal d.o.f.s which are to be eliminated, while the principal ones regarded as a boundary set are to be retained in the concept of sub-structure analysis [4]. Solving for ϕ_s from the upper equation of (7), one has the form

$$\phi_s = -(K_{ss} - \lambda M_{ss})^{-1}(K_{sr} - \lambda M_{sr})\phi_r = -T_{sr}\phi_r, \tag{8}$$

where the transformation matrix T_{sr} is a function of λ and is obtained by the Taylor expansion of matrix $(K_{ss} - \lambda M_{ss})^{-1}$ or

$$T_{sr}(\lambda) = T_{sr}^0 - \lambda\left[U_{sr} - U_{ss}T_{sr}^0\right] - \lambda^2\left[U_{ss}U_{sr} - U_{ss}\cdot\left(U_{ss}T_{sr}^0\right)\right] + O(\lambda^3) + \cdots, \tag{9}$$

and

$$T_{sr}^0 = K_{ss}^{-1}K_{sr}, \tag{10}$$

$$U_{sr} = K_{ss}^{-1}M_{sr} \quad \text{and} \quad U_{ss} = K_{ss}^{-1}M_{ss}.$$

If $T_{sr}(\lambda)$ is truncated up to the first order of the eigenvalue λ of the system, then T_{sr} takes the form

$$T_{sr}^1 = T_{sr}^0 - \lambda\left[-U_{sr} + U_{ss}T_{sr}^0\right], \tag{11}$$

and the secondary co-ordinates ϕ_s satisfy

$$\phi_s \approx -T_{sr}^1\phi_r \quad \text{and} \quad T_{sr}(\lambda) \approx T_{sr}^1. \tag{12}$$

Note that the matrix T_{sr}^0 does not contain any term associated with λ in the upper equation, and is the transformation matrix for traditional static condensations [3, 4]. In addition, since the matrix K is positive definite, sub-matrix K_{ss} is non-singular and thus K_{ss}^{-1} exists, and so do all terms associated with T_{sr}^0 and K_{ss}^{-1}. Substituting equation (8) into the lower equation of (7), it leads to the form

$$[(K_{rr} - \lambda M_{rr}) - (K_{rs} - \lambda M_{rs})T_{sr}]\phi_r = 0. \tag{13}$$

From equations (8), one may write

$$\phi = \begin{Bmatrix} \phi_s \\ \phi_r \end{Bmatrix} = T\cdot\phi_r = \begin{bmatrix} -T_{sr} \\ \mathbf{I} \end{bmatrix}\phi_r, \tag{14}$$

in which T of dimension $n \times r$ is the transformation matrix, and $\mathbf{I}$ the identity matrix of order r. But since T_{sr} is approximated by T_{sr}^1, one has only an approximate $\hat{\phi}$, or

$$\phi \approx \hat{\phi} = \hat{T}\cdot\phi_r = \begin{bmatrix} -T_{sr}^1 \\ \mathbf{I} \end{bmatrix}\phi_r, \tag{15}$$

instead of ϕ. The main function of the transformation matrix in equation (14) or (15) is that it maps the basis ϕ_r of condensed sub-space back into the original modal space ϕ. In other words,

$$T : \phi_r \to \phi \quad \text{and} \quad \hat{T} : \phi_r \to \hat{\phi} \tag{16}$$

with condition $||\phi - \hat{\phi}|| < \varepsilon$, where ε is a small and negligible error parameter, and they are normalized in the same manner.

Using equation (11) into equation (13), expanding the resulting equation, and collecting the terms of like power of λ, one has

$$[\tilde{K}_{rr} - \lambda\tilde{M}_{rr} + \lambda^2\tilde{C}_{rr} + \lambda^3\cdots]\phi_r = 0, \tag{17}$$

where

$$\tilde{K}_{rr} = K_{rr} - K_{rs}T_{sr}^0, \tag{18a}$$

$$\tilde{M}_{rr} = M_{rr} + [-M_{rs} + K_{rs}U_{ss}]\cdot T_{sr}^0 - K_{rs}\cdot U_{sr} \tag{18b}$$

$$\tilde{C}_{rr} = [-M_{rs} + K_{rs}U_{ss}]\cdot[U_{sr} - (U_{ss}\cdot T_{sr}^0)]. \tag{18c}$$

Sub-matrices in equations (18) have been grouped for the sake of the computation. In addition, in case the matrix M is diagonal, matrices $\tilde{M}_{rr}$ and $\tilde{C}_{rr}$ may be further simplified to

$$\tilde{M}_{rr} = M_{rr} + K_{rs}\cdot U_{ss}\cdot T^0_{sr}, \tag{19a}$$

$$\tilde{C}_{rr} = -(K_{rs}\cdot U_{ss})\cdot(U_{ss}\cdot T^0_{sr}). \tag{19b}$$

It is clear that equations (9) and (13) are valid only if the condition [14]

$$||(\mathbf{I} - \lambda U_{ss})^{-1}|| \leq \frac{1}{1 - \lambda||U_{ss}||} \tag{20a}$$

is valid. Or equivalently, the expression

$$\rho_0\left(K_{ss}^{-1}\cdot M_{ss}\right) = \rho_0(U_{ss}) < \tfrac{1}{\lambda} \tag{20b}$$

is satisfied. Here $\rho_0(.)$ denotes the spectral radius or the maximum eigenvalue of the matrix.

Equation (13) becomes dynamic reduction [7] if the terms of order of λ higher than two are neglected. Or, one writes the approximate eigenvalue problem as

$$\left[\tilde{K}_{rr} - \hat{\lambda}\tilde{M}_{rr}\right]\phi_r = 0. \tag{21}$$

However, the derivation of the traditional dynamic reduction method is based on the principle of virtual work, [e.g., see reference 6]. As a consequence, the constraint equation (20) is somehow lost due to the assumption of the dependence by acceleration vector terms. If all sub-mass matrices are further set to zero except M_{rr} in equations (18), then equation (21) may even reduce to the static condensation. Hence, equation (12) or (14) plays the role of linking the secondary (or internal) and primary (or boundary) co-ordinates in the condensation methods.

3. SELECTION OF PRINCIPAL CO-ORDINATES

From the last section, it is obvious that sufficiently accurate retention of the kinetic energy of the system is necessary for condensation. It turns out that the following question arises: How many and which co-ordinates must be chosen as principal d.o.f. in order to ensure sufficiently accurate eigen-frequencies? Traditionally, selection is done mainly by relying on the engineer's experiences based on the dynamic natures of systems. Unfortunately, the determination of principal co-ordinates may not always be obvious for many physical systems. Moreover, the principal co-ordinates depending upon the engineer's intuition are simply not reliable enough. The so-called "automatic" selection method was presented by Henshell and Ong [15]. The method is to scan the leading diagonal terms and to select those degrees that give the highest eigenvalues as the secondary degrees. As far as the author can see, the method is good solely for a few ideal systems. Another selection technique [16], which is based on the Ritz basis vectors, can be taken as a special case of the Gershgorin theorem [11, 17]. Thus, as discussed in reference [11], it is reliable only for cases where the system matrix is diagonally dominant or the Gershgorin disks of the matrix are mutually isolated from others. Otherwise, error arises.

In order to generalize the selection technique based on the Gershgorin disks, a computation procedure, which adopts the QR (or QL) decomposition, has been proposed in reference [12]. In the procedure, a new termination criterion is imposed at the stage of iterative Householder transformation to reduce the system matrix into tri-diagonal form.

Basically, the criterion is accomplished together with an exchange sub-routine reported in reference [18]. The selection scheme is sophisticated if only the eigenvalues are of interest. However, the basis of the transformed space can hardly be traced by the utilization of the exchange procedure. And thus it is limited in use. In fact, some works, [e.g., 15,19–21] have been done in order to overcome the problem of the selection of principal d.o.f.s when using a condensation method. Unfortunately, they gave only a few heuristic guidelines. None of them, to the author's knowledge, can be directly applied in a computer program even though those guidelines are quite useful. Thus, the main goal of this paper is to find a way that may solve this dilemma and to bridge the gap between the guidelines and computer programming.

As it is well known that eigenvalues are close relative to the Rayleigh quotient $\lambda(\phi)$ and there are defined by

$$\lambda = \frac{\phi^{\mathrm{T}} K \phi}{\phi^{\mathrm{T}} M \phi}. \tag{22}$$

If the response vector $\mathbf{u}(t)$ is set as $\mathbf{u}(t) = \phi e^{i\omega t}$ with $\lambda = \omega^2$, then the eigenvalue can be written as

$$\lambda = \frac{V}{\tilde{T}}, \tag{23}$$

where $V = \phi^{\mathrm{T}} K \phi$ is the strain energy and $T = \omega^2 \tilde{T} = \omega^2 \phi^{\mathrm{T}} M \phi$ is kinetic energy corresponding to mode ϕ. Therefore, evaluating the modal energy of individual modes may be a way to solve the problem of identifying primary d.o.f.s.

A generalized eigenvalue problem in equation (5) can be easily transformed to the standard form [1] as

$$A\phi = \lambda \mathbf{I} \phi, \tag{24}$$

for an arbitrary eigenpair (λ, ϕ) for $\lambda \in \Lambda$ and $\phi \in \Phi$. Matrix $A = [a_{ij}]$ in equation (24) is called the system matrix which may be symmetric or non-symmetric depending on the transformation. Therefore, without loss of generality, the discussion henceforth will be given to the standard eigenvalue problem. In the next step, one defines a set of vector $\mathrm{v}_k \subseteq \Re^n$, $k = 1, 2, 3, \ldots, n$, in the neighborhood of the natural basis $\hat{\mathbf{e}}_k = \delta_{kj}$(Kronecker delta, $\delta_{kj} = 1$ if $k = j$ and $\delta_{kj} = 0$ for $k \neq j$), $k = 1, 2, 3, \ldots, n$, such that for some small γ

$$\mathbf{v}_k = \left\{ v_i \middle| \forall v_i \in \mathbf{v}_k, v_i = c\gamma^{|k-i|} \text{ if } |k-i| \leq b; \quad v_i = 0, \text{ otherwise} \right\}^{\mathrm{T}}, \tag{25}$$

where the bandwidth of the system matrix is $2b + 1$. The parameter c in equation (25) is a normalization factor. In fact, $\mathbf{v}_k$ has the form

$$\mathbf{v}_k = \left\{ 0, \ldots 0, \underbrace{\gamma^b, \gamma^{b-1}, \ldots, \gamma}_{b}, \overbrace{1}^{k\text{th}}, \underbrace{\gamma, \gamma^2, \ldots, \gamma^b}_{b}, 0, \ldots, 0 \right\}^{\mathrm{T}} \tag{26}$$

if c has taken the value of unity. That is, the basis $\mathbf{v}_k$ can be expressed in terms of the linear combinations of the natural basis. Note also that if the system matrix is diagonal or $b = 0$, then $\mathbf{v}_k = \hat{\mathbf{e}}_k$ for all k. Clearly, all $\mathbf{v}_k$ are linearly independent if the parameter γ is set to a small value since they are in the vicinity of the natural basis. Figure 1 depicts the relations between the natural basis and $\mathbf{v}_k$ for $n = 3$. Therefore, $\mathbf{v}_k$ may be called a 'pseudo-natural basis' and spans the space of dimension n.

From the physical point of a matrix, the quadratic transformation by expanding $\mathbf{v}_k^{\mathrm{T}} A \ \mathbf{v}_k$ is to consider the adjacent elements of a_{kk} with different weights a_{kk}. And the elements closer to a_{kk} have higher weight. In addition, in case γ is taken as positive, then the effects

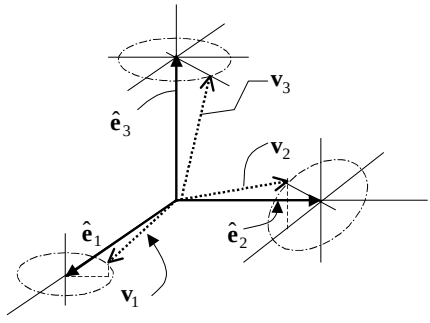

Figure 1. Basis $\mathbf{v}$ is in the vicinity of the natural basis $\hat{\mathbf{e}}$.

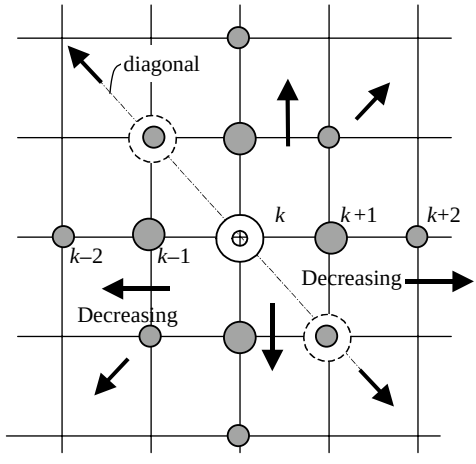

Figure 2. Decreasing weights of the adjacent elements to a_{kk} when a negative γ is chosen.

of the adjacent entries are considered to "flow in" to a_{kk}, while a negative γ means "flowing out" from it. Furthermore, if a larger value of γ is chosen, even more weight has been put on the corresponding diagonal entry. Figure 2 shows the concept. In fact, if one defines an $n \times n$ transformation matrix P by collecting all $\mathbf{v}_k$ as its column vectors, then P is non-singular. And the similarity transformation of A by P is given by

$$B = P^{-1} A P \tag{27}$$

where B preserves the eigenvalues of A, as it is well known. However, the diagonal terms of B and A will not be equal even though $\text{trace}(A) = \text{trace}(B)$ since the transformation in equation (27). Note also that the computation of inverse of P would be impractical if the dimension n is getting large. As the consequence, the transformation of equation (27) does not give much help in the computation.

Let us define a special matrix transpose T^* by an ordinary matrix transpose operation in addition to change of sign in all non-diagonal terms, i.e., to perform a matrix transpose and change the sign of elements except those of the axis of transposing. Therefore operation T^* satisfies $(A^{\mathrm{T}^*})^{\mathrm{T}^*} = A$. Let kth element of vector $\mathbf{v}_k$ be the pivot, then T^*

transpose of the vector as

$$(\mathbf{v}_k)^{\mathrm{T}^*} = \left\{ 0, \ldots 0, \underbrace{-\gamma^b, -\gamma^{b-1}, \ldots, -\gamma}_{b}, \overbrace{1}^{k\text{th}}, \underbrace{-\gamma, -\gamma^2, \ldots, -\gamma^b}_{b}, 0, \ldots, 0 \right\}. \tag{28}$$

Thus, if a matrix Q is collecting all $(\mathbf{v}_k)^{\mathrm{T}^*}$, $k = 1, 2, \ldots, n$, as its column vectors, then it satisfies

$$Q = P^{\mathrm{T}^*} \cong P^{-1} \tag{29}$$

and the transformation in equation (25) may be approximated by

$$B \cong P^{\mathrm{T}^*} A P \tag{30}$$

if γ is taken a small value close to zero. Therefore, the diagonal elements of B contain all eigenvalue information of A, but with a small amount of error that depends on the magnitude of γ.

Define a scalar function $\mathscr{U}$ in the orthonormal basis ϕ of A such that

$$\mathscr{U} = \sum_i \phi_i^{\mathrm{T}} A \phi = \sum_i V_i = \mathrm{trace}(A), \tag{31}$$

is the total strain energy of the system and can be easily computed from A. V_{i} in equation (31) is called the modal energy of the ith mode. Now, let

$$\hat{\mathscr{U}} = \sum_i \frac{\mathbf{v}_i^{\mathrm{T}^*} A \mathbf{v}_i}{\mathbf{v}_i^{\mathrm{T}^*} \cdot \mathbf{v}_i} \cong \sum_i \mathbf{v}_i^{\mathrm{T}^*} A \mathbf{v}_i = \sum_i \hat{V}_i = \mathrm{trace}(B), \tag{32}$$

where $\hat{V}_i$, for all i, are the approximated modal energies in the space spanned by the pseudo-natural basis $\mathbf{v}$. In case the system matrix A is diagonal, then

$$\hat{\mathscr{U}} = \mathscr{U} = \sum_i \hat{\mathbf{e}}_i^{\mathrm{T}} A \hat{\mathbf{e}}_i \tag{33}$$

since $\mathbf{v}$ and ϕ are all equal to the natural basis $\hat{\mathbf{e}}$. And the modal energies V_i are directly mapped into the natural basis. As a consequence, one is able to discard those coordinates with higher value of a_{ii} since they contain more modal strain energy and account for the higher values of eigenvalues. In general, however, matrix A is not diagonal but banded with some bandwidth in most systems. Hence, for any eigenvector ϕ_i of A can be expressed in terms of the linear combinations of $\hat{\mathbf{e}}$, i.e.,

$$\phi_i = \sum_{j=1}^{n} \alpha_j \hat{\mathbf{e}}_j, \tag{34}$$

for all $1 \leqslant i \leqslant n$. However, one may expect that there exists an $\hat{\mathbf{e}}_k$ such that the ϕ_i is just in the neighborhood of that $\hat{\mathbf{e}}_k$. In other words, one expects that most of α_i are close to zero except a few terms. Let us assume ϕ_i be in the vicinity of $\hat{\mathbf{e}}_k$. Then

$$\begin{aligned} &\alpha_k = O(\varepsilon^0), \\ &\alpha_{k-1} = O(\varepsilon^1) = \alpha_{k+1}, \\ &\alpha_{k-2} = O(\varepsilon^2) = \alpha_{k+2}, \\ &\cdots \end{aligned} \tag{35}$$

Therefore, the ith modal energy V_i may be written as

$$\begin{aligned} V_i &= \phi_i^{\mathrm{T}} A \phi_i \\ &\cong (\hat{\mathbf{e}}_k + O(\varepsilon)[\hat{\mathbf{e}}_{k-1} + \hat{\mathbf{e}}_{k+1}] + \cdots)^{\mathrm{T}} A (\hat{\mathbf{e}}_k + O(\varepsilon)[\hat{\mathbf{e}}_{k-1} + \hat{\mathbf{e}}_{k+1}] + \cdots) \\ &= (\hat{\mathbf{e}}_k)^{\mathrm{T}} A \hat{\mathbf{e}}_k + \varepsilon \left(\hat{\mathbf{e}}_k^{\mathrm{T}} A [\hat{\mathbf{e}}_{k-1} + \hat{\mathbf{e}}_{k+1}] + [\hat{\mathbf{e}}_{k-1}^{\mathrm{T}} + \hat{\mathbf{e}}_{k+1}^{\mathrm{T}}] A \hat{\mathbf{e}}_k \right) + \varepsilon^2(\ldots) + \cdots . \end{aligned} \tag{36}$$

Similarly, the corresponding modal energy approximated by $\mathbf{v}$ is

$$\begin{aligned} \hat{V}_k &= (\mathbf{v}_k^{\mathrm{T}^*} A \mathbf{v}_k) \\ &= (\hat{\mathbf{e}}_k)^{\mathrm{T}} A \hat{\mathbf{e}}_k + \gamma \left(\hat{\mathbf{e}}_k^{\mathrm{T}} A [\hat{\mathbf{e}}_{k-1} + \hat{\mathbf{e}}_{k+1}] - [\hat{\mathbf{e}}_{k-1}^{\mathrm{T}} + \hat{\mathbf{e}}_{k+1}^{\mathrm{T}}] A \hat{\mathbf{e}}_k \right) + \gamma^2(\ldots) + \cdots . \end{aligned} \tag{37}$$

By comparing equation (36) with (37), one can conclude that as $\gamma \to \varepsilon$, which is small, there exists an approximate modal energy of mode k such that $\hat{V}_k \to V_i$ with error of order ε. Examining equation (37), one has noticed that there exist negative terms which are produced by the T^* operation. Thus, the approximate modal energy $\hat{V}_k$ largely depends on the system matrix A, as will be discussed in the following.

(i) *Matrix A is symmetric*

In case matrix A is symmetric, equation (37) can be simplified as

$$\hat{V}_k \cong (\hat{\mathbf{e}}_k)^{\mathrm{T}} A \hat{\mathbf{e}}_k - \gamma^2 \left(\hat{\mathbf{e}}_{k-1}^{\mathrm{T}} A [\hat{\mathbf{e}}_{k-1} + \hat{\mathbf{e}}_{k+1}] + \hat{\mathbf{e}}_{k+1}^{\mathrm{T}} A [\hat{\mathbf{e}}_{k-1} + \hat{\mathbf{e}}_{k+1}] \right), \tag{38}$$

where the terms associated with γ^1 cancell out each other because of symmetry. It is clear that the kth modal energy in equation (38) contains two parts: the modal energy produced from the diagonal entry of A and that from the terms in the vicinity. However, the variation of $\hat{V}_k$ mainly depends on the sum in the parentheses, which are the terms associated with γ^2.

Taking the derivative of $\hat{V}_k$ with respect to parameter γ from equation (38), one has

$$\frac{\partial \hat{V}_k}{\partial \gamma} = -2\gamma \left(\hat{\mathbf{e}}_{k-1}^{\mathrm{T}} A [\hat{\mathbf{e}}_{k-1} + \hat{\mathbf{e}}_{k+1}] + \hat{\mathbf{e}}_{k+1}^{\mathrm{T}} A [\hat{\mathbf{e}}_{k-1} + \hat{\mathbf{e}}_{k+1}] \right) + \cdots = -2\gamma \cdot \delta_k + \cdots, \tag{39}$$

where the new parameter δ_k has been used to signify the sum of the terms in the parentheses. Note from equation (39) $\partial \hat{V}_k / \partial \gamma$ is the function of parameter γ and $\partial \lambda_i / \partial \gamma \cong \partial \hat{V}_k / \partial \gamma$, $1 \leqslant i \leqslant n$. On the other hand, for the sake of computations, the derivative may be approximated by

$$s_k = \frac{b_{k,k} - a_{k,k}}{\gamma - 0} \cong \frac{\partial \hat{V}_k}{\partial \gamma} \quad \text{for all } 1 \leq k \leq n, \tag{40}$$

where $b_{k,k}$ is the kth diagonal terms in matrix B of equation (30). The parameter $\mathbf{s}$ is actually the rate change of modal energies with respect to γ, or called the (approximate) modal energy gradients. Figure 3 shows the idea for the case $n = 2$. Note also that $b_{k,k} = \hat{V}_k$. Therefore, the approximate energy gradient vector can be easily computed from the diagonal terms of $b_{k,k}$ and $a_{k,k}$. As a consequence, the modal energy changes for individual modes can be evaluated.

Define a new parameter μ_k, $k = 1, 2, \ldots, n$, such that

$$\mu_k = \frac{1}{\hat{V}_k} \cong \frac{1}{(\hat{\mathbf{e}}_k)^{\mathrm{T}} A \, \hat{\mathbf{e}}_k + 2\gamma \cdot s_k}. \tag{41}$$

Depending on the size of μ_k, which is a function of $a_{k,k}$, parameter γ and the kth modal energy gradient, it may be used to classify the secondary d.o.f.s from the primary ones. In other words, the d.o.f.s with larger μ_k are more likely to be primary. For this reason, parameter μ is called 'the index of classification.' If the variation of $\hat{V}_k$ is increasing, μ_k

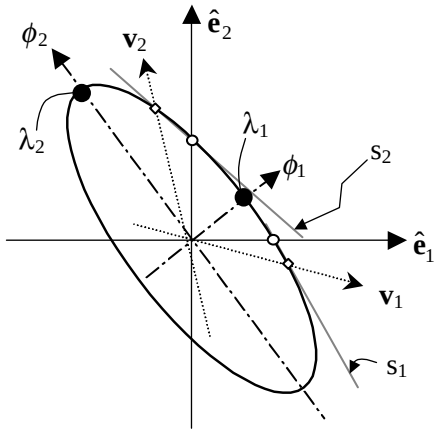

Figure 3. Energy gradients for $n=2$ (●; eigenvalues; ○; $a_{k,k}$; ◇; $b_{k,k}$).

TABLE 1

Tendency for signs of energy gradients and γ of for the kth d.o.f. of symmetric matrices (P: primary, S: secondary)

γ	Sign of s_k	$\hat{V}_k$	kth d.o.f. tends to be
$\gamma<0$	+	Decreasing	P
	—	Increasing	S
$\gamma>0$	—	Decreasing	P
	+	Increasing	S

turns out to decrease. Then the kth d.o.f. tends to be secondary and there is the modal energy flow out from $a_{k,k}$. On the contrary, if $\hat{V}_k$ is decreasing, $\hat{V}_k$ is providing the energy to $a_{k,k}$. Thus, the kth mode has lower modal energy and may be taken as a primary d.o.f. In addition, the magnitudes and signs of γ and s_k decide the tendency of this modal energy variation. All the conclusions are summarized in Table 1.

Therefore, the definition of the primary degrees may be given as those degrees whose diagonal terms of the system matrix obtain modal energy with relatively high variation rate from the neighborhood. While, on the other hand, the secondary degrees are the degrees whose diagonal terms provide their own modal energies with relatively high gradient of energy variations. It also implies that the primary degrees tend to affect the lower eigenvalues, and the secondary may affect mostly the higher.

(ii) *Matrix A is non-symmetric*

In case the system matrix A is non-symmetric but with real eigenvalues, the procedure mentioned earlier has to be modified. From equation (37), one keeps the terms up to the first order of γ, or

$$\hat{V}_k \cong (\hat{\mathbf{e}}_k)^{\mathrm{T}} A\,\hat{\mathbf{e}}_k + \gamma\left(\hat{\mathbf{e}}_k^{\mathrm{T}} A[\hat{\mathbf{e}}_{k-1} + \hat{\mathbf{e}}_{k+1}] - [\hat{\mathbf{e}}_{k-1}^{\mathrm{T}} + \hat{\mathbf{e}}_{k+1}^{\mathrm{T}}] A\,\hat{\mathbf{e}}_k\right). \tag{42}$$

And the derivative with respect to γ has an approximate form

$$s_k \cong \frac{\partial \hat{V}_k}{\partial \gamma} \cong \left(\hat{\mathbf{e}}_k^{\mathrm{T}} A\,[\hat{\mathbf{e}}_{k-1} + \hat{\mathbf{e}}_{k+1}] - [\hat{\mathbf{e}}_{k-1}^{\mathrm{T}} + \hat{\mathbf{e}}_{k+1}^{\mathrm{T}}] A\,\hat{\mathbf{e}}_k\right), \tag{43}$$

where the modal energy gradients s_k is defined the same as equation (41). Unlike its counterpart, the modal energy gradients here is not a function of γ, but only a function of elements of A in the vicinity of $a_{k,k}$. Therefore, the classification for the non-symmetric matrices is more effective than that for symmetric ones. From equations (42) and (43), one can express the approximate modal energy $\hat{V}_k$ as

$$\hat{V}_k \cong (\hat{\mathbf{e}}_k)^{\mathrm{T}} A\, \hat{\mathbf{e}}_k + \gamma \cdot s_k. \tag{44}$$

Therefore, similar to the case of symmetric matrices, one can define

$$\mu_k = \frac{1}{\hat{V}_k} \cong \frac{1}{(\hat{\mathbf{e}}_k)^{\mathrm{T}} A\, \hat{\mathbf{e}}_k + \gamma \cdot s_k} \tag{45}$$

for the index of classification. Meanwhile, one can also summarize the effects as of the modal energy gradients to this index, as shown in Table 1.

Hence, by using the pseudo-natural basis $\mathbf{v}$ and evaluation the approximate modal energy $\hat{V}_k$ and the indices of classification, one is able to discriminate the secondary d.o.f.s as well as the primary. This method is thus named the 'modal energy selection method.'

4. THE MAGNITUDE OF PARAMETER γ

As it has been mentioned in the derivation, the magnitude of parameter γ is directly related to the efficiency of the discrimination of d.o.f.s. In order to decide its magnitude, one may write the transformation matrix P in equation (27) as the identity matrix with a perturbation matrix ΔE, or

$$P = \mathbf{I} - \Delta E. \tag{46}$$

Clearly, the perturbation matrix ΔE is a function of γ and has the norm

$$||\Delta E||_\infty = 2\sum_{i=1}^{b} |\gamma|^i = \frac{2|\gamma|}{1-|\gamma|} \tag{47}$$

if $|\gamma| < 1.0$ and $|\gamma|^b \approx 0$. Hence, the inverse of P can be correspondingly expressed as $P^{-1} = (\mathbf{I} - \Delta E)^{-1}$, or

$$P^{-1} = \mathbf{I} + \Delta E + (\Delta E)^2 + \cdots \tag{48}$$

with condition $||\Delta E|| < 1.0$, or equivalently, $|\gamma| < \frac{1}{3}$. On the other hand, T^* of matrix P or P^{T^*}, may also be written in the perturbation form $P^{\mathrm{T}^*} = \mathbf{I} - \Delta E^*$. It is clear that the perturbation matrix $\Delta E^* = -\Delta E$ since the definition of T^* transpose. Therefore, the error bound for the difference between P^{-1} and P^{T^*} is

$$\begin{aligned} ||P^{-1} - P^{\mathrm{T}^*}|| &= ||(\mathbf{I} + \Delta E + \Delta E^2 + \cdots) - (\mathbf{I} + \Delta E)|| \\ &\leq ||\Delta E||^2 + ||\Delta E||^3 + \cdots \cong \frac{4|\gamma|^3}{1-|\gamma|}. \end{aligned} \tag{49}$$

In case $\gamma = 0{\cdot}2$, for example, the error between them is approximately bounded at the matrix norm 0·04. In other words, using the modal energy selection method (MESM), the decision based on the value below this value is not reliable.

Examining equations (37), (38) and (42), the approximate modal energy is valid only if the terms of higher orders are comparatively smaller than the first order of γ term. That is,

at least the expression

$$\left|\hat{\mathbf{e}}_k^{\mathrm{T}} A[\hat{\mathbf{e}}_{k-1} + \hat{\mathbf{e}}_{k+1}] - [\hat{\mathbf{e}}_{k-1}^{\mathrm{T}} + \hat{\mathbf{e}}_{k+1}^{\mathrm{T}}] A\hat{\mathbf{e}}_k\right| \gg \gamma \cdot \left|\hat{\mathbf{e}}_k^{\mathrm{T}} A[\hat{\mathbf{e}}_{k-2} + \hat{\mathbf{e}}_{k+2}]\right.$$
$$\left. - [\hat{\mathbf{e}}_{k-2}^{\mathrm{T}} + \hat{\mathbf{e}}_{k+2}^{\mathrm{T}}] A\hat{\mathbf{e}}_k - \hat{\mathbf{e}}_{k-1}^{\mathrm{T}} A[\hat{\mathbf{e}}_{k-1} + \hat{\mathbf{e}}_{k+1}] - [\hat{\mathbf{e}}_{k-1}^{\mathrm{T}} + \hat{\mathbf{e}}_{k+1}^{\mathrm{T}}] A\hat{\mathbf{e}}_{k-1}\right| \quad (50)$$

must be true. However, this condition can be satisfied automatically if the tri-diagonal terms are dominant, or all elements of A are in the same order. Refer to Figure 2 for the explanation. Hence, on the other hand, as the guidance of selection for γ, it is imperative that the chosen parameter γ must not change the sign of $\hat{V}_k$ from $a_{k,k}$. Otherwise, the MESM fails. Moreover, from equation (38) and (42), if all $\hat{V}_k$, $k = 1, 2, \ldots, n$, of the matrix are equal, the MESM fails as well.

5. APPLICATION EXAMPLES

In order to demonstrate the procedure of the MESM, two examples are given below. The first one is an arbitrarily given matrix of small dimension, while the second is the simple structural problem formulated from an FEM.

(i) *Example* 1: *A* 5×5 *matrix*

A symmetric matrix given by

$$A = \begin{bmatrix} 4.1 & & & & & 1 \\ -0.3 & 3.0 & & & & 2 \\ 0.0 & 0.4 & 3.0 & & & 3 \\ 0.0 & -0.4 & -0.3 & 3.0 & & 4 \\ 0.0 & 0.0 & 0.3 & 0.2 & 1.9 & 5 \end{bmatrix}$$

is considered as the first example. Note that the numbers to the right of the matrix are d.o.f.s. One cannot identify the effects of each degree using any existing methods since it has three repeated Gershgorin disks at center 3·0, shown in Figure 4. Two eigenvalues in the isolated disks have been known. The one with the center at 1·9 is the primary degree, while that at 4·1 is secondary. However by applying MESM, let $\gamma = 0{\cdot}1$ in equation (26), one is able to define the five pseudo-basis $\mathbf{v}_k$, $k = 1$–5. Therefore, the approximate modal energies and the gradients of the individual d.o.f.s can be calculated from equations (32) and (40), respectively, and the indices of classification are computed in accordance with equation (41). The results of the computation are then shown in Figure 5.

The dashed line in Figure 5 indicates that fifth degree of A is primary since it has the relatively largest μ. The first degree is secondary because μ_1 is the smallest. All other three d.o.f.s cannot be identified by the indices of classification along. However, s_2 and s_5 appear to be negative. Thus, according to the results concluded in Table 1, these two degrees have the tendency to be primary. In other words, the second and fifth affect the lower eigenvalues most. In addition, the figure also shows that the difference between

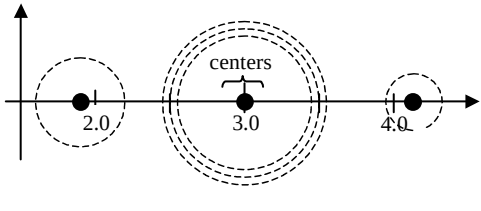

Figure 4. Gershgorin disks of matrix A.

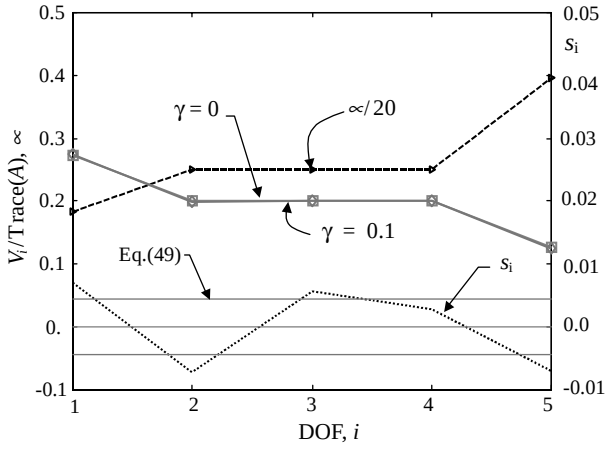

Figure 5. Normalized modal energies of A for example 1 (□; diag(A); ◇; diag(B) $\gamma = 0.1$; ---, μ_i; . . ., modal energy gradients).

transformation of equations (27) (denoted by $\gamma = 0$) and (30) is negligibly small. However, the approximate modal energy gradients defined by equation (40) still reveal the trends of energy variations. The fourth degree is recognized as the secondary; however, the conclusion may not be reliable since the computed energy gradient is less than the bounds defined in equation (49).

In order to verify the results predicted from Figure 5, all eigenvalues are computed by eliminating one degree from matrix A, and the results are plotted in Figure 6. As one can see from Figure 6, the elimination of degree 2 (denoted by $\times$) and 5 (denoted by asterisks) do affect the lowest two eigenvalues that coincide with the predicted conclusion. And the highest eigenvalue deviates from the true value (denoted by •) with large difference if the first degree is eliminated (denoted by $+$). However, the deletion of the first degree hardly affect the lower four eigenvalues.

(ii) *Example* 2: *A prismatic beam*

A prismatic cantilever beam, as shown in Figure 7, is taken as the second demonstration example. The beam is divided into five flexural elements, all of which have the same flexural rigidity EI. A consistent mass matrix is employed in the FE formulation. The assembled system stiffness and mass matrices may be found from lots of texts, e.g., Geradin and Rixen [22] and will not be shown here. In addition, this example is chosen since at least five principal (denoted by odd degrees in Figure 7) and five secondary (denoted by even degrees) d.o.f.s have been known *a priori* from the physical point of view. However, as it will be shown in the next context, one will apply the MESM to select the primary d.o.f.s, instead of depending on the engineers' experiences.

Normally, the stiffness and mass matrices are obtained separately when using a FE formulation. The system matrix A is then computed from a transformation of mass matrix on to the stiffness matrix. Depending on the transformation, the system matrix A may be symmetric or non-symmetric. For example, if one decomposes the mass matrix into the form

$$M = RD^2R^{\mathrm{T}} = (RD)(RD)^{\mathrm{T}},$$

where D is a diagonal matrix containing all eigenvalues of M, then the transformation preserves the symmetry of FEM formulation. Nevertheless, one is able to get two different system matrices that contain the same eigenvalues.

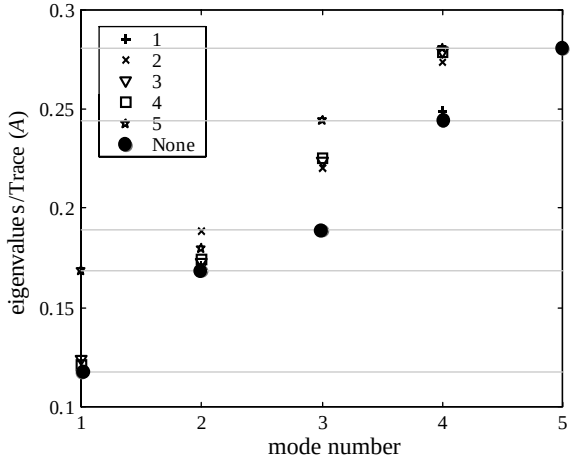

Figure 6. Normalized eigenvalues with different eliminated d.o.f.s.

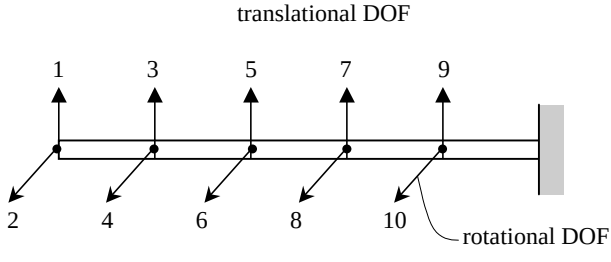

Figure 7. Prismatic beam with 10 d.o.f.s.

In order to apply MESM, let $\gamma = 0.02$ in equation (25), one is able to define the ten pseudo-basis $\mathbf{v}_k$, $k = 1$–10. Therefore, the approximate modal energies of the individual d.o.f.s can be calculated by using equation (32). The results of the computations are shown in Figures 8 and 9 for symmetric and non-symmetric matrices respectively.

It can be clearly seen from Figure 8 that there exist four peaks in the indices of classification μ_i. They happen to be the odd d.o.f.s. Together with the first, there are five primary d.o.f.s. All others are secondary since μ_i are close to zero, for all even i's. The numerical results of several d.o.f.s are shown in Table 2 for comparison. Note also that the error is defined by the vector norm $||\lambda_{true} - \lambda_{cond}||$, where $\lambda \ldots$ denotes the sorted eigenvalues.

On the other hand, the same sign and value of γ are applied to the non-symmetric matrix and the results are shown in Figure 9. Again, there exist peaks and valleys at odd and even d.o.f.s respectively. One immediately concludes that the odd degrees are primary while the evens ones are secondary. This conclusion is exactly the same as the common engineer's senses. In addition to the discrimination of primary d.o.f.s, one also knows that the first degree is the most important primary since it accounts for the largest classification index and the modal energy change. Similarly, the second degree, which appears to have smallest μ, is the one that accounts the most contribution to the highest eigenvalue. Thus, one can

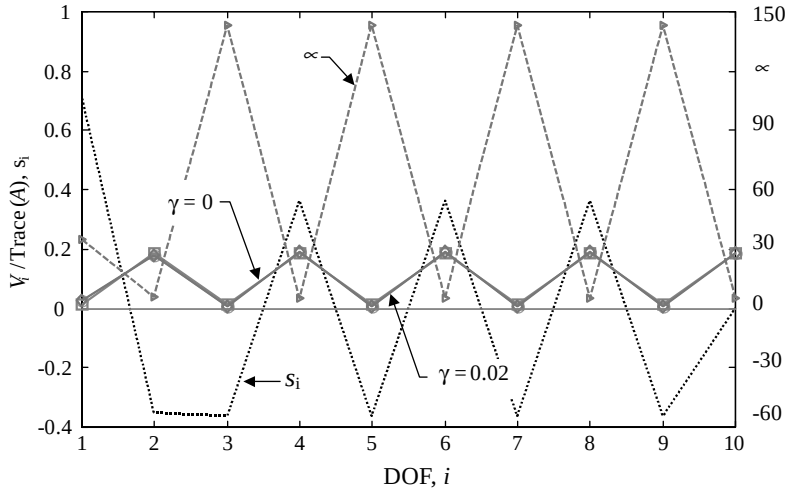

Figure 8. Normalized modal energies for the symmetric A of the example (□,diag(A); ◇, diag(B) with $\gamma = 0.02$; - - -, μ_i; . . ., energy gradients).

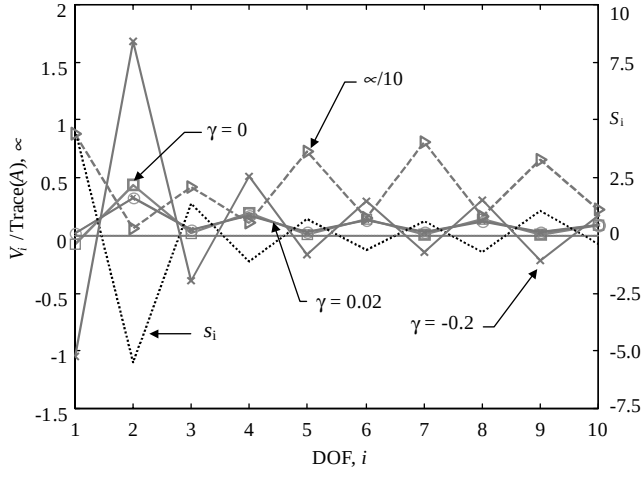

Figure 9. Normalized modal energies for the non-symmetric A of the example (□,diag(A); ◇, diag(B) with $\gamma = 0.02$; ×, diag(B) with $\gamma = -0.2$; - - -, μ_i; . . ., energy gradients).

eliminate this degree first without including much error to the lower eigenvalues. On the other hand, if one does not include the first degree, high error may be expected. All other d.o.f.s except the first and second appear in almost equal order in the classification indices. All these results can be also substantiated by the data shown in Table 3. Note that the data in Table 3 have been normalized by $\mathscr{U}$, or the trace of A.

However, by comparing Figures 8 and 9 for the first d.o.f., one may notice that the contrary for the first d.o.f. even though both predictions are qualitatively similar. It is quite apparent, from equations (39) and (43), that the prediction for the case of non-symmetric matrices is more accurate and thus reliable than that of symmetric one.

It is worth noting that the approximate modal energies for $\gamma = -0.2$ are also plotted for the reason of comparison. As one can see from the figure, there exist several places where

TABLE 2

Normalized eigenvalues obtained by eliminating one degree from symmetric system matrix

Eigenvalues/trace (A)	Eliminated d.o.f.				
	First	Second	Third	Fourth	None
	0·0002	0·0000	0·0004	0·0000	0·0000
	0·0023	0·0008	0·0018	0·0015	0·0001
	0·0090	0·0049	0·0069	0·0045	0·0009
	0·0213	0·0147	0·0200	0·0100	0·0036
	0·0993	0·0266	0·0992	0·0237	0·0098
	0·1397	0·1163	0·1373	0·1256	0·0271
	0·1946	0·1666	0·1922	0·1919	0·0580
	0·2447	0·2213	0·2468	0·2031	0·1219
	0·2746	0·2630	0·2810	0·2540	0·2461
					0·5325
Error	0·1176	0·0432	0·1180	0·0587	0·0

TABLE 3

Eigenvalues obtained by eliminating one degree from non-symmetric system matrix

Eigenvalues/ trace(A)	Eliminated d.o.f.									
	First	Second	Third	Fourth	Fifth	Sixth	Seventh	Eighth	Ninth	Tenth
	0·0001	−0·0252	0·0001	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000
	0·0007	0·0000	0·0007	0·0005	0·0007	0·0002	0·0006	0·0001	0·0003	0·0003
	0·0031	0·0006	0·0025	0·0032	0·0032	0·0015	0·0032	0·0015	0·0021	0·0021
	0·0091	0·0030	0·0084	0·0056	0·0074	0·0085	0·0074	0·0089	0·0080	0·0036
	0·0262	0·0099	0·0266	0·0118	0·0271	0·0107	0·0269	0·0105	0·0259	0·0116
	0·0576	0·0310	0·0575	0·0397	0·0530	0·0525	0·0542	0·0479	0·0579	0·0372
	0·1235	0·0737	0·1170	0·1060	0·1199	0·0809	0·1172	0·0990	0·1209	0·0948
	0·2505	0·1643	0·2407	0·2450	0·2443	0·1956	0·2459	0·1673	0·2432	0·2231
	0·5979	0·3047	0·5267	0·3925	0·5313	0·5039	0·5321	0·5262	0·5323	0·5316
Error	0·3819	0·0783	0·3123	0·1976	0·3181	0·2704	0·3190	0·2876	0·3191	0·3053

the approximate modal energies go below zero in case γ is taken as value of -0.2. Therefore, MESM with $\gamma = -0.2$ is inadequate since there exist sign changes in $\hat{V}_k$ when k is odd.

6. CONCLUDING REMARKS

A new and automatic selection technique based on the approximate modal energy has been developed for matrix condensations in this paper. The method is called the 'modal energy selection method.' By defining a new basis in the neighborhood of the original natural space, the individual modal energy gradients can be estimated. The primary degrees of freedom for the eigenvalue problem are then determined according to the indices of classification and the variation of the energies. In case the energy variation of a degree tends to increase in this neighborhood, that degree is classified as secondary since it relatively has tendency to provide the energy to the nodes nearby. On the other hand, if the energy variation is decreasing, then it is primary. All the classification criteria are finally

mapped to one parameter, which is called the index of classification. That is, by examining the magnitude of these indices, one is able to determine the primary and secondary degrees from the system matrix. Moreover, the error bound is also estimated in this report. The new selection method is demonstrated by a well-known cantilever beam problem in which both primary and secondary degrees are known. In addition to the diagram, numerical data also provided to substantiate the new method.

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